ο υπολογισμός των ολοκληρωμάτων στην πράξη…

1► $\int (6t^2 - 3\eta\mu t - 2\sqrt{t})\,dt = 2t^3 + 3\sigma\upsilon\nu t - \frac{4}{3}t^{\frac{3}{2}} + c$

2► $\int (2^x - 3e^x + 1)\,dx = \frac{2^x}{\ln 2} - 3e^x + x + c$

3► $\int \frac{x^2 + \sqrt{x} - 1}{x^3}\,dx = \int (\frac{1}{x} + x^{-\frac{5}{2}} - x^{-3})\,dx = \ln|x| - \frac{2}{3}x^{-\frac{3}{2}} + \frac{1}{2}x^{-2} + c$

4► $\int (\sigma\upsilon\nu x - x\eta\mu x)\,dx = x\sigma\upsilon\nu x + c$

5► $\int (x+1)e^x\,dx = xe^x + c$

6► $\int \frac{1 - \ln s}{s^2}\,ds = \frac{\ln s}{s} + c$

7► $\int \frac{1}{x\ln^2 x}\,dx = \int \frac{\frac{1}{x}}{\ln^2 x}\,dx = -\frac{1}{\ln x} + c$

8► $\int \frac{\sigma\upsilon\nu x - \eta\mu x}{e^x}\,dx = \int \frac{e^x\sigma\upsilon\nu x - e^x\eta\mu x}{(e^x)^2}\,dx = \frac{\eta\mu x}{e^x} + c$

9► $\int \frac{t^3}{\sqrt{t^4 + 1}}\,dt = \frac{1}{2}\sqrt{t^4 + 1} + c$

10► $\int \frac{3\eta\mu x}{4\sqrt{\sigma\upsilon\nu x}}\,dx = -\frac{3}{2}\sqrt{\sigma\upsilon\nu x} + c$

11► $\int 2xe^{5x^2+1}\,dx = \frac{1}{5}e^{5x^2+1} + c$

12► $\int 3e^x e^{e^x}\,dx = 3e^{e^x} + c$

13► $\int \sigma\upsilon\nu\varphi\,\sigma\upsilon\nu(\eta\mu\varphi)\,d\varphi = \eta\mu(\eta\mu\varphi) + c$

14► $\int (x+1)\eta\mu(x^2 + 2x + 3)\,dx = -\frac{1}{2}\sigma\upsilon\nu(x^2 + 2x + 3) + c$

15► $\int \frac{\sigma\upsilon\nu(\ln\theta)}{\theta}\,d\theta = \eta\mu(\ln\theta) + c$

$(c \in R)$

16► $\int 2\sigma\upsilon\nu^3x\eta\mu x\, dx = -\frac{1}{2}\sigma\upsilon\nu^4x + c$

17► $\int 3x(5x^2+1)^7\, dx = \frac{3}{80}(5x^2+1)^8 + c$

18► $\int s\sqrt[3]{2s^2+3}\, ds = \int s(2s^2+3)^{\frac{1}{3}}\, ds = \frac{3}{16}(2s^2+3)^{\frac{4}{3}} + c$

19► $\int \frac{x^2}{\sqrt[5]{x^3+1}}\, dx = \int x^2(x^3+1)^{-\frac{1}{5}}\, dx = \frac{5}{12}(x^3+1)^{\frac{4}{5}} + c$

20► $\int \frac{t-1}{(t^2-2t+4)^3}\, dt = \int (t-1)(t^2-2t+4)^{-3}\, dt = -\frac{1}{4(t^2-2t+4)^2} + c$

21► $\int \frac{\sqrt[3]{\ln x}}{x}\, dx = \int \frac{1}{x}\ln^{\frac{1}{3}} x\, dx = \frac{3}{4}\ln^{\frac{4}{3}} x + c$

22► $\int \frac{\varepsilon\varphi^3\theta}{\sigma\upsilon\nu^2\theta}\, d\theta = \frac{\varepsilon\varphi^4\theta}{4} + c$

23► $\int \varepsilon\varphi x\, dx = \int \frac{\eta\mu x}{\sigma\upsilon\nu x}\, dx = -\ln|\sigma\upsilon\nu x| + c$

24► $\int \frac{1+\sigma\upsilon\nu x}{x+\eta\mu x}\, dx = \ln|x+\eta\mu x| + c$

25► $\int \frac{1}{u\ln u}\, du = \int \frac{\frac{1}{u}}{\ln u}\, du = \ln|\ln u| + c$

26► $\int \frac{1}{1+e^x}\, dx = \int \frac{1+e^x-e^x}{1+e^x}\, dx = \int \left(1-\frac{e^x}{1+e^x}\right) dx = x - \ln(1+e^x) + c$

27► $\int \sigma\upsilon\nu 3x\eta\mu 2x\, dx = \int \frac{1}{2}(\eta\mu 5x + \eta\mu(-x))\, dx = \int \left(\frac{\eta\mu 5x}{2} - \frac{\eta\mu x}{2}\right) dx = -\frac{\sigma\upsilon\nu 5x}{10} + \frac{\sigma\upsilon\nu x}{2} + c$

28► $\int \sigma\upsilon\nu^2 x\, dx = \int \frac{1+\sigma\upsilon\nu 2x}{2}\, dx = \int \left(\frac{1}{2} + \frac{\sigma\upsilon\nu 2x}{2}\right) dx = \frac{x}{2} + \frac{\eta\mu 2x}{4} + c$

29► $\int \eta\mu^3 x\, dx = \int \eta\mu^2 x\eta\mu x\, dx = \int (1-\sigma\upsilon\nu^2 x)\eta\mu x\, dx = \int (\eta\mu x - \sigma\upsilon\nu^2 x\eta\mu x)\, dx = -\sigma\upsilon\nu x + \frac{\sigma\upsilon\nu^3 x}{3} + c$

30► $\int \sigma\upsilon\nu^4 x\, dx = \int \left(\frac{1+\sigma\upsilon\nu 2x}{2}\right)^2 dx = \int \left(\frac{1}{4} + \frac{\sigma\upsilon\nu 2x}{2} + \frac{1+\sigma\upsilon\nu 4x}{8}\right) dx = \ldots = \frac{3x}{8} + \frac{\eta\mu 2x}{4} + \frac{\eta\mu 4x}{32} + c$

$(c \in R)$

31▶ $\int ημ^2xσυν^2x\,dx = \int \frac{ημ^2 2x}{4}dx = \int \frac{1-συν4x}{8}dx = \frac{x}{8} - \frac{ημ4x}{32} + c$

32▶ $\int ημ^3xσυν^2x\,dx = \int (1-συν^2x)συν^2xημx dx = \int (συν^2xημx - συν^4xημx)dx = -\frac{συν^3x}{3} + \frac{συν^5x}{5} + c$

33▶ $\int ημ^2xσυν^4xdx = \int ημ^2xσυν^2xσυν^2xdx = \int \frac{ημ^2 2x}{4}\frac{1+συν2x}{2}dx = \int (\frac{ημ^2 2x}{8} + \frac{ημ^2 2xσυν2x}{8})\,dx$

$= \int (\frac{1-συν4x}{16} + \frac{ημ^2 2xσυν2x}{8})\,dx$

$= \frac{x}{16} - \frac{ημ4x}{64} + \frac{ημ^3 2x}{48} + c$

34▶ $\int εφ^2x\,dx = \int (1+εφ^2x-1)dx = εφx - x + c$

35▶ $\int \frac{1}{ημ^2xσυν^2x}dx = \int \frac{ημ^2x+συν^2x}{ημ^2xσυν^2x}dx = \int (\frac{1}{συν^2x} + \frac{1}{ημ^2x})\,dx = εφx - σφx + c$

36▶ $\int \frac{1}{ημx}dx = \int \frac{ημ^2\frac{x}{2} + συν^2\frac{x}{2}}{2ημ\frac{x}{2}συν\frac{x}{2}}dx = \int (\frac{ημ\frac{x}{2}}{2συν\frac{x}{2}} + \frac{συν\frac{x}{2}}{2ημ\frac{x}{2}})\,dx = -\ln\left|συν\frac{x}{2}\right| + \ln\left|ημ\frac{x}{2}\right| + c = \ln\left|εφ\frac{x}{2}\right| + c$

αλλιώς: $= \int \frac{1}{2ημ\frac{x}{2}συν\frac{x}{2}}dx = \int \frac{συν\frac{x}{2}}{ημ\frac{x}{2}}\frac{1}{2συν^2\frac{x}{2}}dx = \int \frac{1}{εφ\frac{x}{2}}(εφ\frac{x}{2})'\,dx = \ln\left|εφ\frac{x}{2}\right| + c$

αλλιώς: αργότερα!

(c∈R)

37▶ $\int \frac{\gamma}{\alpha x+\beta}dx = \frac{\gamma}{\alpha}\ln|\alpha x+\beta| + c$

38▶ $\int \frac{x+1}{x^2+2x+3}dx = \frac{1}{2}\ln|x^2+2x+3| + c$

39▶ $\int \frac{3x^3-x^2+3x-3}{x^4+3x^2}dx = \int \frac{3x^3-x^2+3x-3}{x^2(x^2+3)}dx$

Θα βρω α,β,γ,δ ∈R τέτοιους ώστε $\forall x \in R^*$, να ισχύει:

$$\frac{3x^3-x^2+3x-3}{x^2(x^2+3)} = \frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma x+\delta}{x^2+3} \quad (1)$$

(1) ⇔ $3x^3-x^2+3x-3 = \alpha x(x^2+3)+\beta(x^2+3)+(\gamma x+\delta)x^2$, $\forall x \in R^*$

⇔ $3x^3-x^2+3x-3 = (\alpha+\gamma)x^3+(\beta+\delta)x^2+3\alpha x+3\beta$

$$\Leftrightarrow \begin{cases} \alpha+\gamma = 3 \\ \beta+\delta = -1 \\ 3\alpha = 3 \\ 3\beta = -3 \end{cases} \Leftrightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \\ \gamma = 2 \\ \delta = 0 \end{cases}$$

$$= \int \left(\frac{1}{x}-\frac{1}{x^2}+\frac{2x}{x^2+3}\right)dx$$

$$= \ln|x| + \frac{1}{x} + \ln(x^2+3) + c$$

40▶ $\int \frac{6x^4-3x^3-5x^2-2x+1}{2x^2-x-1}dx =$

$$\begin{array}{r|l} 6x^4-3x^3-5x^2-2x+1 & 2x^2-x-1 \\ \hline -6x^4+3x^3+3x^2 & 3x^2-1 \\ -2x^2-2x+1 & \\ 2x^2-x-1 & \\ -3x & \end{array}$$

$6x^4-3x^3-5x^2-2x+1 = (2x^2-x-1)(3x^2-1) - 3x$

$$\frac{6x^4-3x^3-5x^2-2x+1}{2x^2-x-1} = 3x^2-1-\frac{3x}{2x^2-x-1}$$

Θα βρω α,β ∈R τέτοιους ώστε ∀x∈R / {-1/2,1}, να ισχύει:

$$\frac{3x}{(x-1)(2x+1)} = \frac{\alpha}{x-1}+\frac{\beta}{2x+1} \quad (1)$$

(1) ⇔ $3x = \alpha(2x+1)+\beta(x-1)$, ∀x∈R / {-1/2,1}

⇔ $3x = (2\alpha+\beta)x+\alpha-\beta$

$$\Leftrightarrow \begin{cases} 2\alpha+\beta = 3 \\ \alpha-\beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 1 \\ \beta = 1 \end{cases}$$

$$= \int \left(3x^2-1-\frac{3x}{2x^2-x-1}\right)dx$$

$$= \int \left(3x^2-1-\frac{3x}{(x-1)(2x+1)}\right)dx$$

$$= \int \left(3x^2-1-\frac{1}{x-1}-\frac{1}{2x+1}\right)dx$$

$$= x^3-x-\ln|x-1|-\frac{1}{2}\ln|2x+1|+c$$

(c∈R)

41► $\int \frac{x}{\sigma\upsilon\nu^2 x}dx = \int x(\varepsilon\varphi x)' dx = x\varepsilon\varphi x - \int (x)' \varepsilon\varphi x\, dx = x\varepsilon\varphi x + \ln|\sigma\upsilon\nu x| + c$

42► $\int \eta\mu x\sigma\upsilon\nu x e^{\eta\mu x}\, dx = \int \eta\mu x\left(e^{\eta\mu x}\right)' dx = (\eta\mu x)e^{\eta\mu x} - \int (\eta\mu x)' e^{\eta\mu x} dx = (\eta\mu x)e^{\eta\mu x} - e^{\eta\mu x} + c$

43► $\int \ln^2 t\, dt = \int (t)'\ln^2 t dt = t\ln^2 t - \int t(\ln^2 t)' dt$

$= t\ln^2 t - \int t\frac{1}{t}2\ln t\, dt$

$= t\ln^2 t - \int (t)' 2\ln t\, dt$

$= t\ln^2 t - \left(2t\ln t - \int t(2\ln t)' dt\right)$

$= t\ln^2 t - 2t\ln t + 2t + c$

44► $\int \eta\mu(\ln x)\, dx = \int (x)' \eta\mu(\ln x)\, dx = x\eta\mu(\ln x) - \int x(\eta\mu(\ln x))' dx$

$= x\eta\mu(\ln x) - \int x\, \sigma\upsilon\nu(\ln x)\frac{1}{x}dx$

$= x\eta\mu(\ln x) - \int (x)' \sigma\upsilon\nu(\ln x)\, dx$

$= x\eta\mu(\ln x) - [\, x\sigma\upsilon\nu(\ln x) - \int x(\sigma\upsilon\nu(\ln x))' dx\,]$

$= x\eta\mu(\ln x) - x\sigma\upsilon\nu(\ln x) + \int x(-\eta\mu(\ln x))\frac{1}{x} dx$

$= x\eta\mu(\ln x) - x\sigma\upsilon\nu(\ln x) - \int \eta\mu(\ln x)\, dx$

$\Leftrightarrow \quad 2\int \eta\mu(\ln x)\, dx = x\eta\mu(\ln x) - x\sigma\upsilon\nu(\ln x) + c_1$

$\Leftrightarrow \quad \int \eta\mu(\ln x)\, dx = \frac{x\eta\mu(\ln x) - x\sigma\upsilon\nu(\ln x)}{2} + c$

$(c_1, c \in R)$

45▶ $\int x\sqrt{x+1}\,dx$

Θετω: $u=\sqrt{x+1}$
τότε: $x=u^2-1$
και $dx=2udu$

$$=\int (u^2-1)u2u\,du=\int (2u^4-2u^2)du$$
$$=\frac{2u^5}{5}-\frac{2u^3}{3}+c$$
$$=\frac{2(x+1)^2\sqrt{x+1}}{5}-\frac{2(x+1)\sqrt{x+1}}{3}+c$$

46▶ $\int x^2(x-1)^7\,dx$

Θέτω: $u=x-1$
τότε: $x=u+1$
και $dx=du$

$$=\int (u+1)^2u^7\,du=\int (u^9+2u^8+u^7)\,du$$
$$=\frac{u^{10}}{10}+\frac{2u^9}{9}+\frac{u^8}{8}+c$$
$$=\frac{(x-1)^{10}}{10}+\frac{2(x-1)^9}{9}+\frac{(x-1)^8}{8}+c$$

47▶ $\int \frac{e^{2x}}{1+e^x}dx$

Θέτω: $u=e^x$
οπότε: $du=e^xdx$

$$=\int \frac{u}{u+1}du=\int \frac{u+1-1}{u+1}du=\int \left(1-\frac{1}{u+1}\right)du$$
$$=u-\ln|u+1|+c$$
$$=e^x-\ln(1+e^x)+c$$

48▶ $\int \frac{\sigma\upsilon\nu x}{\eta\mu^2x+3\eta\mu x+2}dx$

Θέτω: $u=\eta\mu x$
οπότε: $du=\sigma\upsilon\nu x dx$

Θα βρω $\alpha,\beta\in R$, τέτοιους ώστε $\forall u\in R\ /\ \{-2,-1\}$, νάναι:

$$\frac{1}{(u+1)(u+2)}=\frac{\alpha}{u+1}+\frac{\beta}{u+2}\quad(1)$$

$(1)\Leftrightarrow 1=\alpha(u+2)+\beta(u+1)$

$\Leftrightarrow 1=(\alpha+\beta)u+2\alpha+\beta$

$\Leftrightarrow \begin{cases}\alpha+\beta=0\\ 2\alpha+\beta=1\end{cases}\Leftrightarrow\begin{cases}\alpha=1\\ \beta=-1\end{cases}$

$$=\int \frac{1}{u^2+3u+2}du=\int \frac{1}{(u+1)(u+2)}du=\int \left(\frac{1}{u+1}-\frac{1}{u+2}\right)du$$
$$=\ln|u+1|-\ln|u+2|+c$$
$$=\ln(1+\eta\mu x)-\ln(2+\eta\mu x)+c$$
$$=\ln\frac{1+\eta\mu x}{2+\eta\mu x}+c$$

$(c\in R)$

49▶ $\int \frac{1}{ημx}dx = \int \frac{ημx}{ημ^2x}dx = \int \frac{ημx}{1-συν^2x}dx$

Θέτω: u=συνx
οπότε: du= -ημxdx

Θα βρω α,β∈R ,τέτοιους ώστε
∀u∈R / {-1,1} να είναι:

$$\frac{1}{(u-1)(u+1)} = \frac{α}{u-1} + \frac{β}{u+1} \quad (1)$$

(1) ⇔ 1=α(u+1)+β(u-1)
⇔ 1=(α+β)u+α-β

$$⇔ \begin{cases} α+β=0 \\ α-β=1 \end{cases} ⇔ \begin{cases} α=1/2 \\ β=-1/2 \end{cases}$$

$$= \int -\frac{1}{1-u^2}du = \int \frac{1}{(u-1)(u+1)}du$$

$$= \int \left(\frac{1/2}{u-1} - \frac{1/2}{u+1}\right)du$$

$$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c$$

$$= \frac{1}{2}\ln\frac{1-συνx}{1+συνx} + c$$

■ $ημx = \dfrac{2εφ\frac{x}{2}}{1+εφ^2\frac{x}{2}}$ ■ $συνx = \dfrac{1-εφ^2\frac{x}{2}}{1+εφ^2\frac{x}{2}}$ ■ $εφx = \dfrac{2εφ\frac{x}{2}}{1-εφ^2\frac{x}{2}}$ ■ $σφx = \dfrac{1-εφ^2\frac{x}{2}}{2εφ\frac{x}{2}}$

η «καθολική αντικατάσταση»: $u = εφ\frac{x}{2}$ οπότε: $du = \frac{1}{2}(1+εφ^2\frac{x}{2})dx$ δηλαδή: $dx = \frac{2}{1+u^2}du$

και: $ημx = \frac{2u}{1+u^2}$, $συνx = \frac{1-u^2}{1+u^2}$, $εφx = \frac{2u}{1-u^2}$, $σφx = \frac{1-u^2}{2u}$

μετατρέπει (με πολύ κόπο πολλές φορές !) τα τριγωνομετρικά ολοκληρώματα σε ρητά

αλλιώς: $\int \frac{1}{ημx}dx$

Θέτω $u = εφ\frac{x}{2}$

οπότε $du = \frac{1}{2}(1+εφ^2\frac{x}{2})dx$

δηλαδή $dx = \frac{2}{1+u^2}du$

και $ημx = \frac{2u}{1+u^2}$

$$= \int \frac{1}{\frac{2u}{1+u^2}} \frac{2}{1+u^2}du$$

$$= \int \frac{1}{u}du$$

$$= \ln|u| + c$$

$$= \ln\left|εφ\frac{x}{2}\right| + c$$

(c∈R)

50► $\int \frac{1}{4\eta\mu x+3\sigma\upsilon\nu x+5}dx$

Θέτω: $u=\varepsilon\varphi\frac{x}{2}$

οπότε: $du=\frac{1}{2}(1+\varepsilon\varphi^2\frac{x}{2})\,dx$

δηλαδή: $dx=\frac{2}{1+u^2}du$

και $\eta\mu x=\frac{2u}{1+u^2}$

$\sigma\upsilon\nu x=\frac{1-u^2}{1+u^2}$

$$=\int \frac{1}{4\frac{2u}{1+u^2}+3\frac{1-u^2}{1+u^2}+5}\frac{2}{1+u^2}du=\int\frac{2}{8u+3-3u^2+5+5u^2}du$$

$$=\int\frac{1}{u^2+4u+4}du$$

$$=\int\frac{1}{(u+2)^2}du$$

$$=-\frac{1}{u+2}+c$$

$$=-\frac{1}{\varepsilon\varphi\frac{x}{2}+2}+c \qquad (c\in R)$$

51► $\int_0^1 x(3x^2+1)^5\,dx=[\frac{(3x^2+1)^6}{36}]_0^1=\frac{4^6}{36}-\frac{1}{36}=\frac{4^6-1}{36}$

52► $\int_0^1 x^2e^{x^3+1}\,dx=[\frac{e^{x^3+1}}{3}]_0^1=\frac{e^2}{3}-\frac{e}{3}=\frac{e(e-1)}{3}$

53► $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \eta\mu^2x\sigma\upsilon\nu x\,dx=[\frac{\eta\mu^3x}{3}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{\eta\mu^3\frac{\pi}{2}}{3}-\frac{\eta\mu^3(-\frac{\pi}{2})}{3}=\frac{2}{3}$

54► $\int_2^3 \frac{3x^4-3x^3+x^2-x-1}{x^2-x}dx$

$3x^4-3x^3+x^2-x-1$	x^2-x
$-3x^4+3x^3$	$3x^2+1$
x^2-x-1	
$-x^2+x$	
-1	

Θα βρω α,β $\in R$ τέτοιους ώστε

$\forall x\in[2,3]$ νάναι: $\frac{-1}{x(x-1)}=\frac{\alpha}{x}+\frac{\beta}{x-1}$ (1)

(1) $\Leftrightarrow -1=\alpha(x-1)+\beta x$

$\Leftrightarrow -1=(\alpha+\beta)x-\alpha$

$\Leftrightarrow \begin{cases}\alpha+\beta=0\\ -\alpha=-1\end{cases} \Leftrightarrow \begin{cases}\alpha=1\\ \beta=-1\end{cases}$

$$=\int_2^3(3x^2+1+\frac{-1}{x(x-1)})\,dx=\int_2^3(3x^2+1+\frac{1}{x}-\frac{1}{x-1})\,dx$$

$$=[x^3+x+\ln x-\ln(x-1)]_2^3$$

$=27+3+\ln3-\ln2-(8+2+\ln2-\ln1)$

$=20+\ln3-2\ln2$

$=20+\ln\frac{3}{4}$

55► $\int_0^{\frac{\pi}{2}} e^x συν2x\,dx = \int_0^{\frac{\pi}{2}} (e^x)' συν2x\,dx = [e^x συν2x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2e^x ημ2x\,dx$

$= [e^x συν2x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2e^x)' ημ2x\,dx$

$= [e^x συν2x]_0^{\frac{\pi}{2}} + [2e^x ημ2x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 4e^x συν2x\,dx$

$\Leftrightarrow \quad 5\int_0^{\frac{\pi}{2}} e^x συν2x\,dx = [e^x συν2x]_0^{\frac{\pi}{2}} + [2e^x ημ2x]_0^{\frac{\pi}{2}}$

$\Leftrightarrow \quad \int_0^{\frac{\pi}{2}} e^x συν2x\,dx = \frac{e^{\frac{\pi}{2}} συνπ - e^0 συν0 + 2e^{\frac{\pi}{2}} ημπ - 2e^0 ημ0}{5} = \frac{-e^{\frac{\pi}{2}} - 1}{5}$

56► $\int_0^1 x\sqrt{1-x}\,dx$

Θέτω: $u = \sqrt{1-x} = \begin{cases} 0 & \text{για } x=1 \\ 1 & \text{για } x=0 \end{cases}$

οπότε: $x = 1-u^2$

και $dx = -2udu$

$= \int_1^0 -(1-u^2)u2u\,du = \int_1^0 (2u^4 - 2u^2)\,du = [\frac{2u^5}{5} - \frac{2u^3}{3}]_1^0$

$= 0 - 0 - (\frac{2}{5} - \frac{2}{3})$

$= \frac{4}{15}$

57► $\int_0^1 x^2\sqrt{4-x^2}\,dx$

Θέτω: $x = 2ημu,\ u \in [0, \frac{\pi}{6}]$

οπότε: $dx = 2συνu\,du$

για $x=1$: $ημu = \frac{1}{2} \Leftrightarrow u = \frac{\pi}{6}$

για $x=0$: $ημu = 0 \Leftrightarrow u = 0$

$= \int_0^{\frac{\pi}{6}} 4ημ^2u\sqrt{4-4ημ^2u}\ 2συνu\,du = \int_0^{\frac{\pi}{6}} 16ημ^2u συν^2u\,du$

$= \int_0^{\frac{\pi}{6}} 4ημ^2 2u\,du$

$= \int_0^{\frac{\pi}{6}} 2(1 - συν4u)\,du$

$= [2u - \frac{ημ4u}{2}]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

58▶ αν $f(x) = \begin{cases} x , & -\pi \le x \le \pi \\ \eta\mu x , & x > 0 \end{cases}$ η f είναι συνεχής στο [-π, π] αφού:

η f είναι συνεχής στα [-π, 0) και (0, π] και $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0) = 0$ (δηλ. η f συνεχής και στο 0)

συνεπώς: $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx = \int_{-\pi}^{0} xdx + \int_{0}^{\pi} \eta\mu x dx = \ldots = 2 - \frac{\pi^2}{2}$

59▶ αν f περιττή , τότε: $\int_{-\alpha}^{\alpha} f(x)dx = \int_{-\alpha}^{0} f(x)dx + \int_{0}^{\alpha} f(x)dx$

Θέτω: u = -x = { για x=0 → 0, για x=-α → α }

οπότε: x = -u
και dx = -du

$= \int_{\alpha}^{0} -f(-u)du + \int_{0}^{\alpha} f(x)dx$

$= \int_{\alpha}^{0} f(u)du + \int_{0}^{\alpha} f(x)dx$ (γιατί αφού η f είναι περιττή ισχύει: f(-u) = -f(u))

$= -\int_{0}^{\alpha} f(u)du + \int_{0}^{\alpha} f(x)dx$

$= 0$

ομοίως:

αν f περιττή, τότε: $\int_{-\alpha}^{\alpha} f(x)dx = 2\int_{0}^{\alpha} f(x)dx$

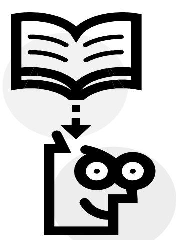
60▶ αν $I_\nu = \int_0^{\frac{\pi}{2}} \eta\mu^\nu x dx$, $\nu \in N^*$ τότε $\forall \nu > 2$: $I_\nu = \frac{\nu - 1}{\nu} I_{\nu-2}$, αφού:

$I_\nu = \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-1} x \eta\mu x dx = \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-1} x(-\sigma\upsilon\nu x)' dx = [-\eta\mu^{\nu-1} x \sigma\upsilon\nu x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (\nu - 1)\eta\mu^{\nu-2} x \sigma\upsilon\nu^2 x dx$

$= 0 + (\nu-1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x (1 - \eta\mu^2 x) dx$

$= (\nu-1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x dx - (\nu-1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu} x dx$

$= (\nu-1) I_{\nu-2} - (\nu-1) I_\nu$



$\Leftrightarrow \quad I_\nu + (\nu-1)I_\nu = (\nu-1)I_{\nu-2}$

$\Leftrightarrow \quad I_\nu = \frac{\nu - 1}{\nu} I_{\nu-2}$